# Unidimensional modellization of dynamic footing behaviour

## Modélisation unidimensionnelle du comportement dynamique d'une fondation superficielle

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SYNOPSIS A physical one-dimensional model is presented to reproduce the dynamic behaviour of a circular rigid footing resting on a semi-infinite medium. In this simplified approach, the semi-infinite medium is replaced by a solid of increasing section resting on a single degree-of-freedom system. After checking the agreement between the discrete model and the homogeneous elastic medium, the behaviour of a heterogeneous medium is investigated. A practical application of the model besides footing analysis is given.

INTRODUCTION

The dynamic behaviour of a circular footing resting on top of an homogeneous elastic half-space has been investigated by Reissner (1936), Quinlan (1953), Sung (1953) and Hiesh (1962). For a rigid footing the vertical displacement functions were found by Bycroft (1956) whereas Lysmer (1965) introduced a single degree of freedom system which has practically the same behaviour. For harmonic behaviour, one defines the uniform vertical movement u of the footing with (fig. 1.a)

$$u = u_o \cdot e^{i\omega t} = \frac{Q_o\ e^{i\omega t}}{GR}(f_1 + if_2) \qquad (1)$$

$u_o$: amplitude of the harmonic movement
$\omega$ : circular frequency of the movement
t : time
$Q_o$: amplitude of force
G : shear modulus of the elastic half-space
$f_1$, $f_2$ : displacement functions

The Lysmer analog is governed by the following equation : (fig. 1.c.)

$$m\frac{\partial^2 u}{\partial t^2} + \frac{3.4\ R^2}{1-\nu}\sqrt{\rho G}\ \frac{\partial u}{\partial t} + \frac{4GR}{1-\nu}u = Q \quad (2)$$

in which : m : mass of the footing
R : radius of the circular footing
$\nu$ : Poisson's ratio of the elastic medium
$\rho$ : the mass density of the medium
Q : $Q_o\ e^{i\omega t}$ the harmonic vertical force

This analog has brought a clear simplification of the behaviour of a rigid footing but its generalisation to non-homogenous and non-elastic media is not straightforward. Funston and Hall (1967) have considered a hyperbolic tangent law for the static reaction and have deduced equivalent moduli and damping constants.

In order to introduce the effects of heterogeneity and of non-linearity, it is best to model the behaviour of the medium locally. Finite element methods can be used but a much simpler approach can be investigated, based on the concept of the equivalent solid.

THE EQUIVALENT SOLID

Though primitive, the concept of the equivalent solid (Pauw 1954) can now be pursued again, thanks to the reference solution given by Lysmer. Since we shall assume that the solid deforms vertically in order to obtain a one-dimensional model in the vertical direction, the parameters defining the equivalent solid are (fig. 1.b.) :

$\bar{m}$ : the mass attached to the footing
$\bar{\rho}$ : the mass density of the solid
$E_v$: the vertical modulus of deformation
$\Omega(z)$: the horizontal section of the solid as a function of the depth; or its radius r(z)
H : the height of the solid
$k_H$: the spring constant at the base of the model
$C_H$: the damping constant at the base of the model

The model will conform to the following conditions :

- exact static deflection
- adequate harmonic behaviour of a mass less disk
- adequate harmonic behaviour with mass attached to the footing

As the number of conditions is smaller than the unkown constants and functions, we shall keep constant some the physical properties of the model :

$\bar{m}$ = m, the real mass of the footing
$\bar{\rho}$ = $\rho$, the mass density of the medium
H should be arbitrarily chosen, but $k_H$ and $C_H$ would be determined therefrom.

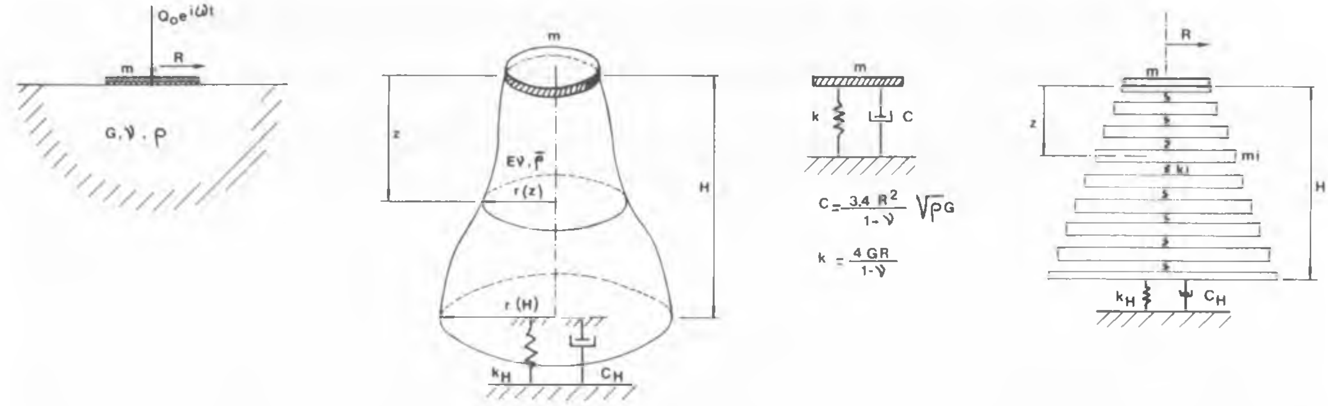


Figure 1 :
a) Elastic half-space b) Equivalent Solid c) Lysmer's Analog d) Discretized model

From these assumptions, the main unknowns are $\Omega(z)$ or $r(z)$ and $E_v$.

For the static deflection of the footing under load Q, we assume :

$$s_\infty = s_H + \Delta s \qquad (3)$$

with $s_\infty = \frac{Q\,(1-\nu)}{4\ GR}$ the static deflection of the footing on the half-space

$s_H = \frac{Q\,(1-\nu)}{4G\ r(H)}$ the static deflection of the rigid base of the solid resting on the half-space below depth H

$\Delta s = \int_0^H \frac{Q\ dz}{\Omega(z)\ E_v}$ the compression of the equivalent solid

Writing the conditon (3) for any depth under an incremental form, since the displacement is uniform through any horizontal section of the solid, we get :

$$s_z = s_{z+dz} + \frac{Q\ dz}{\Omega(z)\ E_v}$$

which can be expressed by :

$$\frac{1-\nu}{4GR} = \frac{1-\nu}{4G\ (r+dr)} + \frac{dz}{E_v\ \pi r\ (r+dr)}$$

from which we obtain a very simple expression :

$$\frac{dr}{dz} = \frac{4G}{(1-\nu)\ E_v \cdot \pi} = \frac{E}{E_v} \cdot \frac{2}{\pi\ (1-\nu^2)} \qquad (4)$$

We deduce from this expression that the equivalent solid is a frustrum of a cone whose current radius is given by :

$$r(z) = R + az \qquad (5)$$

with : $a = \frac{E}{E_v}\ \frac{2}{\pi\ (1-\nu^2)}$

For the dynamic behaviour, we assume that the mechanical impedance of a current section be equal to the damping of Lysmer's analog at that level :

$$\pi r^2\ \sqrt{E_v \rho} = \frac{3.4\ r^2}{1-\nu}\ \sqrt{\rho G} \qquad (6)$$

This condition can be transformed into :

$$E_v = \frac{G}{0.85\ (1-\nu)^2} \qquad (7)$$

Substituting (7) in (4), one obtains :

$$a = \frac{dr}{dz} = \frac{1-\nu}{\sqrt{0.85}} \qquad (8)$$

which means that the opening angle of the equivalent frustum of a cone depends only on the value of ν. In principle for the elastic case, the lower boundary of the solid can be chosen at any depth, provided it rests on its own Lysmer's analog defined by :

$$k_H = \frac{4Gr(H)}{1-\nu} \qquad (9)$$

$$C_H = \frac{3.4\ r^2(H)}{1-\nu}\ \sqrt{\rho G} \qquad (10)$$

HOMOGENEOUS ELASTIC CASE

The behaviour of the equivalent one-dimensional model can be computed using a lumped parameter approach and integrating explicitly the equations of movement, using a sufficiently short time interval Δt. The cone is replaced by n+1 masses and n springs obtained from its discretization in the vertical direction :

$$m_i = \frac{\pi r^2(z_i)\ \rho H}{n} \qquad i=0 \text{ to } n \qquad (11)$$

$$k_i = E_v\ \frac{\pi r^2\ (z_{i+\frac{1}{2}})\ n}{H} \qquad i=1 \text{ to } n \qquad (12)$$

The mass at the top is equal to the sum of the mass of the foundation and $m_o/2$ whereas the mass at the lower boundary is equal to $m_n/2$.

The parameters of the problem solved have been chosen so as to obtain easily adimensional data:

with $v_s$ : shear wave velocity
$a_o$ : adimensional frequency
$B_z$ : adimensional mass parameter

$$\left.\begin{array}{r} \nu = 1/3 \\ G = 5\ \text{MPa} \\ R = 1\ \text{m} \\ Q = 30\ \text{MN} \\ \rho = 0.002\ \text{Mkg/m}^3 \end{array}\right] \rightarrow \left[\begin{array}{l} v_s = \sqrt{G/\rho} = 50\ \text{m/s} \\ s_\infty = 1\ \text{m} \\ a_o = \frac{\omega R}{v_s} = \frac{\omega\ [\text{rad/s}]}{50} \\ B_z = \frac{1-\nu}{4}\ \frac{m}{\rho R^3} = \frac{m\ [\text{Mkg}]}{0.012} \end{array}\right.$$

The force signal is applied to the top mass of the model so that transient behaviour can be handled and the response displacement is computed. For a periodic force signal, the integration is conducted until the movement becomes periodic and its amplitude is obtained. The results which are compared to the Lysmer solution in figure 2 have been obtained from such a procedure on a solid with a thickness of 14 m and divided into 28 elements. The time increment (1 ms) was less than a quarter of the critical time interval. The agreement between the magnification curves computed for various mass ratio by both methods is very satisfactory.

## GENERALIZATION OF THE MODEL

Since a one-dimensional model has been found that matches the behaviour of the homogeneous elastic medium, one is tempted to generalize the field of application by introducing local properties of the subsoil. It seems indeed reasonable to assume that the heterogeneity presented by a layered medium can be accounted for by setting the constants of the lumped parameters according to the properties of the layers at each depth. For an elastic layered medium, values of the masses and spring constants are then computed with (11) and (12), replacing $\rho$ by the local value of the mass density, $E_v$ by the value obtained by (7) with local parameters and $r(z)$ by the value obtained by integration over depth of equation (4).

Some degree of approximation is involved in this straightforward generalization. It is clear that a one-dimensional model will not reproduce in detail all aspects of phenomena involving the 4 types of waves (P,S,R,L) taking place in an elastic layered half-space. To show a limitation of the model, an infinitely contrasted heterogeneity is considered below, for which a solution exists. Other types of generalization (non-elastic medium, visco-elastic medium) are presented elsewhere (Holeyman (1984)), but comparison with exact solutions is prevented because they do not exist.

The case of the elastic layer resting on an incompressible basis is the most stringent test to which the proposed model can be subjected. The massless solution has been determined by Bycroft (1956) whereas Warburton (1957) has considered the influence of the mass m vibrating with the rigid disk. In the case of the elastic layer of finite thickness H, the model height is limited to this thickness and its base is fixed because $k_H$ and $C_H$ given by (9) and (10) become infinite.

For the static solution, the displacement $\Delta s$ is obtained from :

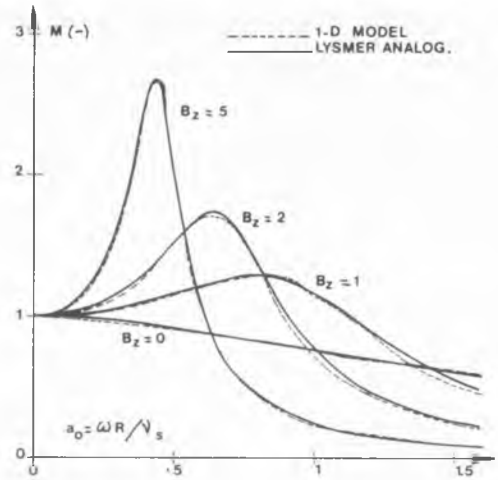


Figure 2 : Dynamic Magnification Curves

$$\Delta s = \frac{qR}{E_v a} \int_0^H \frac{dz}{(1+az/R)^2} = \frac{qR}{E_v a}\left(1 - \frac{1}{1+Ha/R}\right) \qquad (13)$$

with $q = Q/\pi R^2$

Expression (13) is compared in fig. 3 (dotted lines for $\nu$=0 and $\nu$=0.5) with the graphical representation of the exact solution produced by Davis (1968) for various values of Poisson's ratio. One notes that despite the gross modelization of the elastic layer, the agreement with the exact solution is good within 10 % for $R/H > 1$. On the contrary, for $R/H < 0.5$, substantial errors occur due the fact that the lateral confinement is not effectively represented in the model.

For the dynamic case, an approximate reference solution is provided by Warburton (1957) with the assumption of a hyperbolic contact stress distribution law. Values of the resonance frequencies and of the amplitude at resonance are given by this author for $\nu$=1/4 for various values of the adimensional mass ratio : $b = m/\rho R^3$. The problem that was solved by the equivalent solid with a fixed basis showed resonance frequencies in approximate concordance with values presented by Warburton, but with

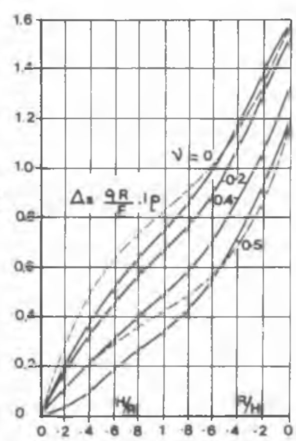


Figure 3 : Deflection on finite elastic layer.

amplitudes higher than the reference amplitudes. This different behaviour of the model comes from its inability to dissipate energy in the radial direction and in this extreme case, energy could not be dissipated in the vertical direction either, due to the fixed end condition. More reasonable values are to be expected if the contrast of the heterogeneity is not infinite. This example demonstrates however that the simple model suggested is not suited to the determination of the amplitude around resonance in elastic finite media.

Fortunately, this is not often the case in soil mechanics as :

- the contrast in heterogeneity of modules of deformation is always finite,
- the medium to be modelled is always a half-space,
- energy is intrinsically dissipated in the soil.

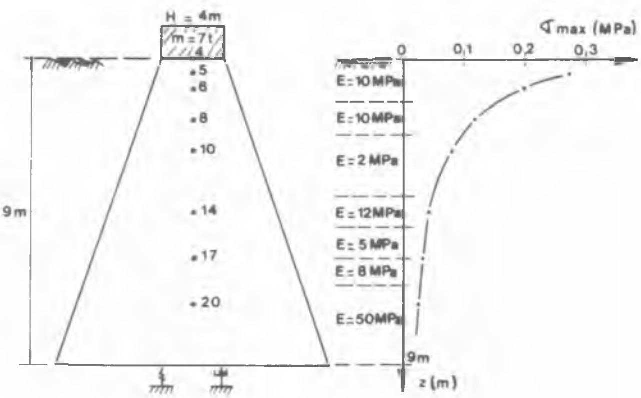


Figure 4 a) Model b) Dynamic stress vs. depth

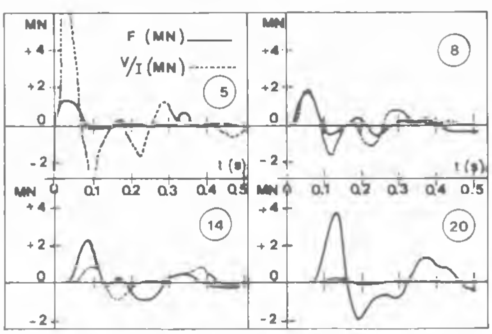


Figure 5 a) Force and velocity at various depths

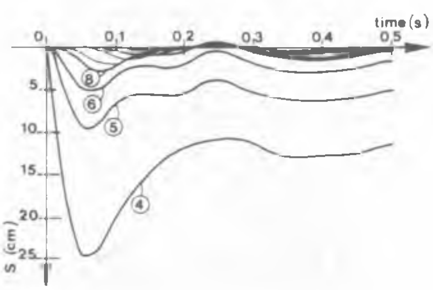


Figure 5 b) Displacements at various depths

TRANSIENT NON-LINEAR LOADING

To illustrate this aspect, a hyperbolic law has been adopted to represent the intrinsic behaviour of the model in the vertical direction. The sollicitation is provided by a falling mass as used in the heavy tamping technique for compaction of soil masses. The mass of the cylindrical tamper is 7000 kg whereas its radius is 0.9 m. The profile is described by the following characteristics, with $q_{ult}$: the ultimate stress used for the hyperbolic law.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| depth of layer [m] | 1.3 | 2.3 | 4.1 | 5.1 | 5.9 | 6.5 | 20 |
| $E_v$ [MPa] | 10 | 10 | 2 | 12 | 5 | 8 | 50 |
| $q_{ult}$ [MPa] | 0.35 | 0.5 | 0.1 | 0.6 | 0.25 | 0.4 | 2.5 |

The mass is dropped from an effective height of 4 m. The computed velocity is the initial condition, while the remaining degrees of freedom are initially at rest. The system is limited to a depth of 9 m, such that its base rests in the last layer considered (see fig. 4a).

The results of such a computation are illustrated by the following diagrams :

- velocity and force signals at various levels (fig. 5a)
- displacement signals at various levels (fig. 5b)
- maximum transient stress as a function of depth (fig. 4b)

The first signals are relevant for the assessment of the displacements and frequencies involved in waves transmitted to neighbouring structures. The second ones are useful for the prediction of the volume of the craters and the last one can be interpreted as a preconsolidation pressure when estimating settlements after treatment of the ground.

CONCLUSIONS

The suggested one-dimensional model can reproduce faithfully the behaviour of a rigid circular foundation resting on an elastic homogeneous half space. The agreement with Lysmer's analog is obtained from a rational formulation of the boundary conditions of the model. This model is apt to take into account non homogeneities of a horizontally layered medium and non-linearities. The example of the elastic layer of a finite thickness illustrates the limitation of the model around resonance. Since damping is intrinsically provided by soils, this model represents a good aproximation for the field of soil mechanics, specially for transient signals, as illustrated by a sample problem relating to heavy tamping.

REFERENCE

Holeyman, A. (1984), "Contribution à l'étude du comportement dynamique non-linéaire des pieux lors de leur battage. Doctoral Thesis, Université Libre de Bruxelles, ICC, July 84, 584 p.