CAVITY EXPANSION ANALYSIS IN ELASTIC – BRITTLE PLASTIC ROCK MASS

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ABSTRACT

A comprehensive approach is presented for the analyses of both cylindrical and spherical cavity expansion in an infinite elastic – brittle plastic rock mass. The rock mass obeys the nonlinear generalized "Hoek – Brown" (H-B) failure criterion which is expressed in a scaled form. A plastic flow rule characterized by a constant dilatancy angle ψ is adopted. Closed form solutions are presented for the extent of the plastic region and the distribution of radial and circumferential stresses. For displacement field, solutions in the plastic region are developed based on small strain theory. Finally, the solutions are validated using finite element method.

Keywords: Hoek – Brown Failure Criterion, Cylindrical and Spherical Cavity Expansion, Elastic – Brittle Plastic Post Failure, Small Strain Theory.

1. INTRODUCTION

The cavity expansion theory has been widely used to solve many problems in geotechnical engineering such as interpretation of in-situ tests, predicting the end bearing and shaft capacity of driven piles. Moreover, an important class of civil engineering problems deals with cavities in rock masses and most practical applications still consider rock as a soil and predict their behavior by using soil-related failure criteria such as the linear Mohr-Coulomb (M-C) criterion. Furthermore, extensive literature [1] is available on cavity analysis in a medium consisting of M-C material. However, the non-linear Hoek-Brown (H-B) [2] failure criterion more rigously represents rock mass behavior. Since its introduction in 1980's, this criterion has been mostly used in cavity contraction analyses for tunneling applications. In this light, Brown et al. [3] presented a closed form solution for the cavity unloading problem in an elastic – brittle plastic material as well as an elastic strain softening plastic material obeying the 1980 H-B failure criterion version [4]. Carranza-Torres and Fairhurst [5] studied also the elasto-plastic response of underground excavation in rock masses obeying the 1997 H-B failure criterion version [6] in which the excavation process is treated as a uniform reduction of internal pressure in symmetrically loaded cylindrical and spherical cavities. In addition to the published closed form expressions, they provided a dimensional graphical representation of their solutions that allows direct estimates of the response of excavations.

However, to the authors' knowledge, not much has been published for cavity expansion problem in a material obeying the H-B failure criterion. In this paper, both cylindrical and spherical cavities in elastic – brittle plastic H-B material are considered. To simplify the governing equations of the problem, the generalized H-B failure criterion expression [2] is used in a "scaled"/non-dimensional form. The use of this scaled form leads to considerable simplification in analyzing the elasto-brittle plastic response of the rock.

This paper starts with a brief recall of the generalized H-B failure criterion and the adopted normalization method. Then, analytical solutions are derived and validated by Finite Element Method (FEM).

2. H-B FAILURE CRITERION

The H-B failure criterion [2] is an empirical criterion developed through curve-fitting of triaxial test data. This criterion assumes isotropic rock and should only be applied to rock masses in which there is a sufficient number of closely spaced discontinuities. In other worlds, the H-B failure criterion is valid for intact rocks or heavily jointed rock masses (i.e. sufficient dense and randomly distributed joints). The latest version of the H-B criterion [2] is defined by:

$$\sigma_1' = \sigma_3' + \sigma_{ci}(m_b \sigma_3'/\sigma_{ci} + s)^a \quad (1)$$

here σ_1' and σ_3' denote, respectively, the major and the minor principal stresses at failure. σ_{ci} is the uniaxial compressive strength of the intact rock. The parameters m_b, s and a describe the rock mass characteristics and depend on the Geotechnical Strength Index GSI, the disturbance factor D and the the intact frictional strength component m_i. They are calculated as:

$$m_b/m_i = e^{\left(\frac{GSI-100}{28-14D}\right)};\ s = e^{\left(\frac{GSI-100}{9-3D}\right)};\ a = 0.5 + \left(e^{\left(-\frac{GSI}{15}\right)} - e^{\left(-\frac{20}{3}\right)}\right)/6, \quad (2)$$

The H-B failure criterion expression (Eq. (1)) defines a relationship between minor and major principal stresses depending on four independent parameters, which can be reducing to a single one using a "scaled" form of the criterion. The suggested transformation involves dividing Eq. (1) by $\left(\sigma_{ci} m_b^{\beta}\right)$ and adding the term $\left(s/m_b^{\beta/a}\right)$ to both sides. With these manipulations, the H-B failure criterion expression becomes:

$$\sigma_1'/(\sigma_{ci} m_b^{\beta}) + s/m_b^{\beta/a} = \sigma_3'/(\sigma_{ci} m_b^{\beta}) + s/m_b^{\beta/a} + \left(\sigma_3'/(\sigma_{ci} m_b^{\beta}) + s/m_b^{\beta/a}\right)^a;\ \beta = \frac{a}{1-a} \quad (3)$$

Thus, the scaled (non-dimensional) minor and major principal stresses are defined naturally as,

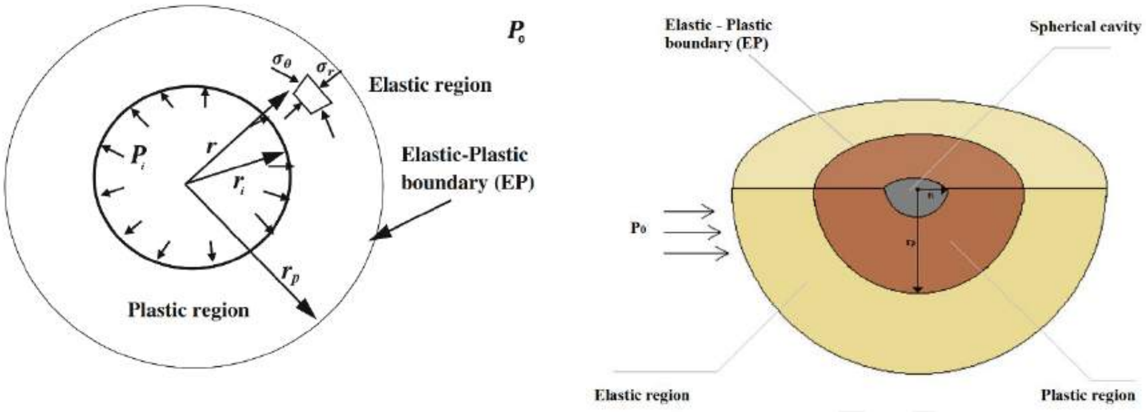
$$\sigma_j^* = \sigma_j'/(\sigma_{ci} m_b^{\beta}) + s/m_b^{\beta/a}; \qquad j = \{1,3\} \quad (4)$$

From now on, normalized stresses will have an asterisk as superscript. For sake of brevity, the term "scaled" will be mostly dropped when referring to a scaled variable unless stated otherwise. When expressed in terms of σ_1^* and σ_3^*, the H-B failure criterion permits a simplified and normalized treatment of the rock mass failure condition. Thanks to such a scaling, the H-B failure criterion is formally simplified as follows:

$$\sigma_1^* = \sigma_3^* + (\sigma_3^*)^a \quad (5)$$

3. PROBLEM STATEMENT

Both cylindrical and spherical cavities in elastic-brittle-plastic H-B material are considered. The geometry of the problem and the boundary conditions are as shown in Figure 1. Let r_i be the internal radius of the cavity and P_0 the far field radial pressure. Let P_i be the internal pressure applied on cavity wall that increases monotonically from its initial value P_0. As the internal pressure P_i increases, the rock mass will initially behave in an elastic manner, until reaching a yield pressure P_y. When the internal pressure P_i exceeds P_y, a plastic region will start spreading from r_i to the "plastic" radius r_p. The remainder of the domain ($r \geq r_p$) belongs to the elastic region.



(a) Cylindrical cavity (b) Spherical cavity

***Figure 1**: Geometry of the problem and boundary conditions*

It should be emphasized that as soon as the materials yields, the peak strength parameters m_b^{peak} and s^{peak} and the peak deformation modulus E^{peak} drop to residual strength parameters m_b^{res} and $s^{res}$and residual deformation modulus E^{res}. Therefore, two cases can be distinguished:

- If residual strength parameters are equal to the peak parameters, then we are defining an "ideally" elastic-plastic material (cf. Figure 2).
- If not, Hoek [7] said that they, rock engineering community, do not have good models to describe this post failure behavior but he suggested two post failure: Elastic-brittle (cf. Figure 2) and strain softening as a starting point. The latter one is beyond the scope of this paper.

To ensure that closed form solution can be obtained, it is necessary to further assume that after yield, the strength of rock drops suddenly to its residual values. Note that the disturbance factor D can be used to achieve a strength and modulus reduction after failure. It is found that $D = 0.7$ is appropriate in most cases [7]. In what follows, only brittle plastic post failure is considered. The perfectly plastic model is simply a limiting case of the brittle one.

The analytical study described hereinafter is conducted based on the scaled form of the H-B failure criterion and pressures and stresses are scaled using residual strength parameters reflected by 'res' as a superscript. On the other hand, the equation of equilibrium for the cavity problem is expressed in terms of radial and circumferential stresses (scaled) as:

$$d\sigma_r^{*res}/dr + k(\sigma_r^{*res} - \sigma_\theta^{*res})/r = 0 \qquad (6)$$

here k is a coefficient equal to 1 in case of a cylindrical cavity and it is equal to 2 in case of a spherical one. Note that major and minor principal stresses are assumed to be equal to radial and circumferential stresses, respectively, i.e. $\sigma_1 = \sigma_r$ and $\sigma_3 = \sigma_\theta$.

4. STRESS AND DIPLACEMENT FIELD – ANALYTICAL SOLUTION

When the internal pressure P_i exceeds P_y, a plastic region starts spreading from r_i to the plastic radius r_p. The remainder of the domain ($r \geq r_p$) belongs to the elastic region. The latter is first studied before investigating the plastic region but firstly, the yield pressure expression is provided hereinafter.

4.1. Yield Pressure

When the internal pressure P_i applied on the cavity wall reaches the yield pressure P_y, the stress of the rock mass at cavity face will satisfy the failure criterion expressed as a function of the peak strength parameters. Thus, the net yield pressure $\Delta P_y^{*peak} = \left(P_y^{*peak} - P_0^{*peak}\right)$ should satisfy the following equation:

$$\Delta P_y^{*peak} = -\Delta P_y^{*peak}/k + \left[P_0^{*peak} - \Delta P_y^{*peak}/k\right]^a \qquad (7)$$

The superscript '$peak$' in the above equation means that pressures are scaled using peak strength parameters m_b^{peak} and s^{peak}. The yield pressure depends only on the far field pressure P_0^{*peak} and the exponent a. The above equation provides an explicit solution when $a = 0.5$:

$$\Delta P_{y\,|a=0.5}^{*peak} = k\left(\sqrt{4P_0^{*peak}(k+1)^2 + 1} - 1\right)/(2(k+1)^2) \qquad (8)$$

When $a \neq 0.5$, there is no explicit solution and Eq. (7) should be solved numerically. The evolution of P_y^{*peak} as a function of P_0^{*peak} is plotted in Figure 3. As can be seen, when $P_0^{*peak} \to 0$, P_y^{*peak} approaches $2P_0^{*peak}$ for cylindrical cavity and approaches $3P_0^{*peak}$ for the spherical one. Secondly, the yield pressure is relatively constant when $GSI > 30$ since the exponent $a \cong 0.5$. Since the governing equations of the problem are scaled using residual strength parameters, the following relationship relating P_y^{*res} to P_y^{*peak} should be used once P_y^{*peak} is found per Eq.(7)

$$P_y^{*res} = \left(m_b^{peak}/m_b^{res}\right)^{\beta} P_y^{*peak} + \left(m_b^{res}\right)^{-\beta}\left(s^{res}/m_b^{res} - s^{peak}/m_b^{peak}\right) \qquad (9)$$

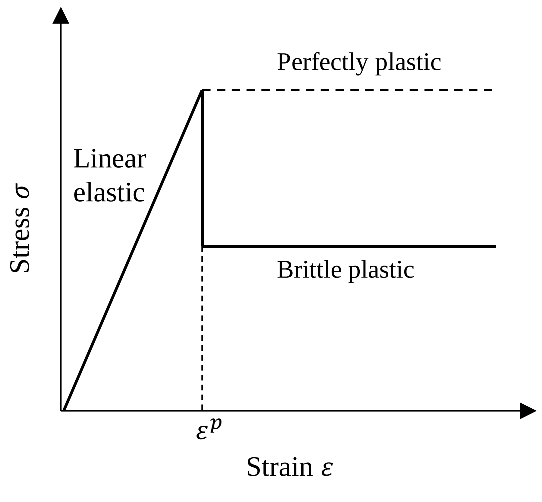


***Figure 2:** Perfectly plastic and brittle plastic post failure.*

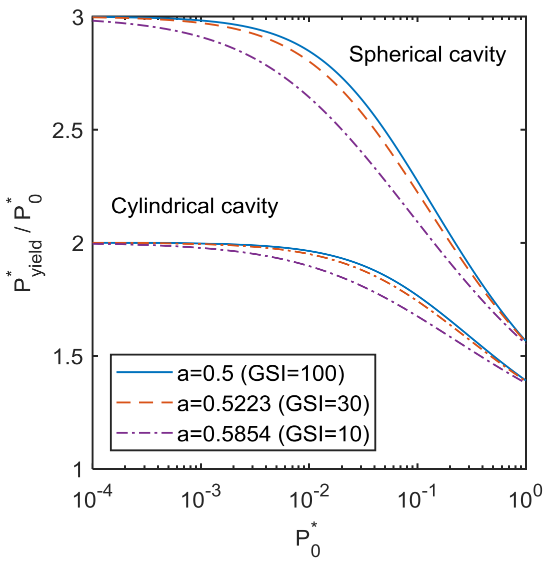


***Figure 3** Scaled yield pressure.*

4.2. Elastic Region $(r \geq r_p)$

Since the solution for this region is well known, only, main results are presented hereinafter. By considering the boundary conditions $\sigma_r^{*res}(r = r_p) = P_y^{*res}$ and $\sigma_r^{*res}(r = \infty) = P_0^{*res}$, radial and circumferential stresses are expressed as:

$$\sigma_r^{*^{res}} - P_0^{*^{res}} = -k(\sigma_\theta^{*^{res}} - P_0^{*^{res}}) = \left(P_y^{*^{res}} - P_0^{*^{res}}\right)\left(r_p/r\right)^{k+1} \tag{10}$$

On the other hand, total strains are written as functions of elastic and plastic strains as follows:

$$\varepsilon_r = \varepsilon_r^e + \varepsilon_r^p \quad ; \quad \varepsilon_\theta = \varepsilon_\theta^e + \varepsilon_\theta^p \tag{11}$$

where ε_r^p and ε_θ^p denote, respectively, the radial and the circumferential plastic strains. They will be evaluated in the next subsection whereas ε_r^e and ε_θ^e are, respectively, the radial and the circumferential elastic strains given as:

$$\varepsilon_r^e = \left(\left(1 + \nu(k-2)\right)\left(\sigma_r^{*^{res}} - P_0^{*^{res}}\right) - k\nu\left(\sigma_\theta^{*^{res}} - P_0^{*^{res}}\right)\right)/(2I_r(1+\nu)^{k-1}) \tag{12}$$

$$\varepsilon_\theta^e = \left((1-\nu)\left(\sigma_\theta^{*^{res}} - P_0^{*^{res}}\right) - \nu\left(\sigma_r^{*^{res}} - P_0^{*^{res}}\right)\right)/(2I_r(1+\nu)^{k-1}) \tag{13}$$

in which, ν is the Poisson's ratio. I_r is the rigidity index of the rock mass and it is expressed according to the region state. When the elastic region prevails which is the case here, I_r is expressed as a function of the peak shear modulus as $I_r = G^{peak}/(\sigma_{ci}(m_b^{res})^\beta)$

by substituting Eqs. (10) into Eqs. (12) and (13) and by taking into account that infinitesimal strain can be written in terms of radial displacement u as $\varepsilon_r = -\frac{du}{dr}$ and $\varepsilon_\theta = -\frac{u}{r}$, the radial displacement can be evaluated as:

$$u = u_{EPB}\left(r_p/r\right)^k \quad ; \quad u_{EPB} = r_p\left(P_y^{*^{res}} - P_0^{*^{res}}\right)/(2\mathrm{k}I_r) \tag{14}$$

here u_{EPB} is the radial displacement at the Elastic – Plastic Boundary (EPB).

4.3. Plastic Region ($\boldsymbol{r_i \le r \le r_p}$)

In this section, analytical solution for the extent of the plastic region and the related stresses and displacement fields are investigated.

4.3.1. Stress field

Substituting the scaled H-B failure criterion expression (Eq. (5)) in the equilibrum equations (Eq. (6)) results in the following differential equation of the circumferential stress:

$$d\sigma_\theta^{*^{res}}/dr + d\left(\sigma_\theta^{*^{res}}\right)^a/dr + k\left(\sigma_\theta^{*^{res}}\right)^a/r = 0 \tag{15}$$

The general solution of the above first-order nonlinear differential equation can be written as:

$$\sigma_\theta^{*^{res}}(r) = \left[aW_0\left(C_1 r^{-\frac{k}{\beta}}\right)\right]^{\beta/a} \tag{16}$$

where C_1 *is* a constant. W_0 is the 0th branch of the Lambert W-function (Omega function). Lambert W-function is the solution of the equation $x = W(x)e^{W(x)}$. To simplify the expressions, the change of variable $R = C_1\, r^{-\frac{k}{\beta}}$ will be used which allow the circumferential stress and radial stress (Eq. (5)) to be reduced to

$$\sigma_r^{*^{res}}(R) = [aW_0(R)]^{\beta/a} + [aW_0(R)]^\beta \quad ; \quad \sigma_\theta^{*^{res}}(R) = [aW_0(R)]^{\beta/a} \tag{17}$$

Based on the hypothesis of the continuous radial stress at the (EPB) and knowing that an internal pressure $P_i^{*^{res}}$ is exerted at the cavity wall, i.e.,

$$P_y^{*^{res}} = \left[aW_0\left(R_p\right)\right]^{\beta/a} + \left[aW_0\left(R_p\right)\right]^\beta \,;\, P_i^{*^{res}} = [aW_0(R_i)]^{\beta/a} + [aW_0(R_i)]^\beta \tag{18}$$

the plastic radius r_p can be calculated once Eqs. (18) is solved for $R_p = C_1\, r_p^{-\frac{k}{\beta}}$ and $R_i = C_1 r_i^{-\frac{k}{\beta}}$. Thus, the normalized plastic radius is simply evaluated as $r_p/r_i = \left(R_i/R_p\right)^{\frac{\beta}{k}}$.

When $a = 0.5$, a closed form solution of the plastic radius can be provided as:

$$(r_p/r_i)_{a=0.5} = \left(\left(\sqrt{4P_i^{*res}+1}-1\right)e^{\sqrt{4P_i^{*res}+1}-1}\right)^{\frac{1}{k}}\left(\left(\sqrt{4P_{y|a=0.5}^{*res}+1}-1\right)e^{\sqrt{4P_{y|a=0.5}^{*res}+1}-1}\right)^{-\frac{1}{k}} \quad (19)$$

When $a \neq 0.5$, there is no closed form solution for the plastic radius and Eqs. (18) should be solved numerically.

The distrubtion of radial and circumferential stresses are shown in Figure 4 for both cylindrical and spherical cavities. As can be seen, a discontinuty due to the brittle charachter is evidenced for the circumferential stress.

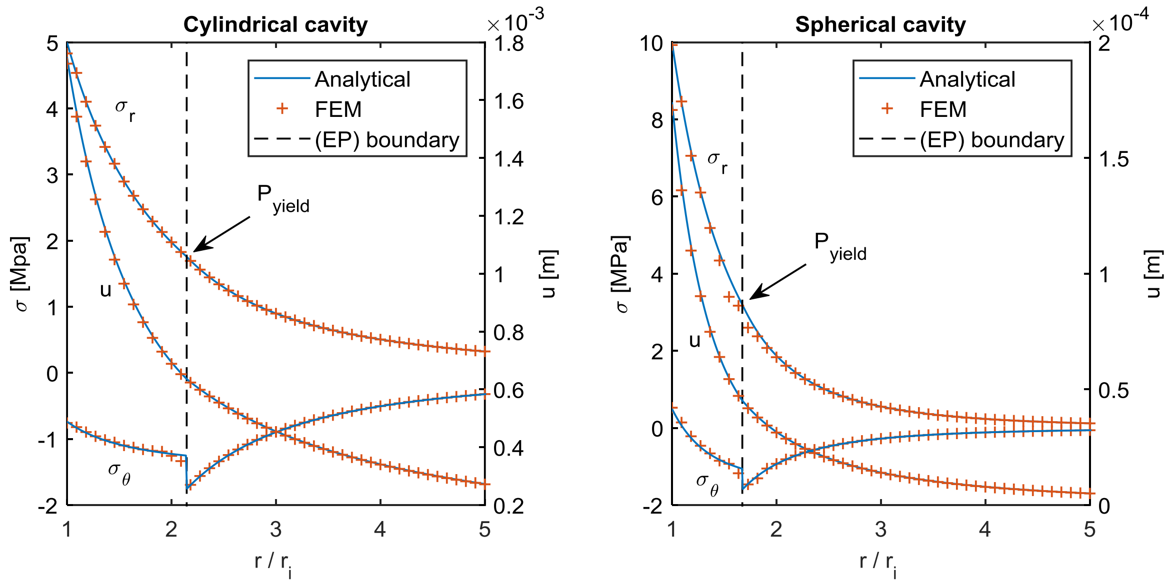


Figure 4: *Distribution of radial and circumferential stresses and displacement for both cylindrical and spherical cavities in a H-B material with* $\sigma_{ci} = 35\ [MPa], GSI = 80, m_i = 4, P_0 = 0, MR = \frac{E_i}{\sigma_{ci}} = 250, \nu = 0.3$ *and* $\psi = 0°$

4.3.2. Displacement field

To determine the displacement field in the plastic zone, a plastic flow rule is needed. Thus, a non-associated flow rule with a constant dilatancy angle ψ is adopted as

$$\varepsilon_r{}^p + k\omega\varepsilon_\theta{}^p = 0 \quad ; \quad \omega = \frac{1-\sin(\psi)}{1+\sin(\psi)} \quad (20)$$

which can be further expressed per Eqs. (11) as:

$$\varepsilon_r + k\omega\varepsilon_\theta = \varepsilon_r{}^e + k\omega\varepsilon_\theta{}^e \quad (21)$$

On the other hand, infinitesimal strains are expressed in terms of radial displacement u as follows:

$$\varepsilon_r = -du/dr \quad ; \quad \varepsilon_\theta = -u/\mathrm{r} \quad (22)$$

Substituting the above equations into Eq. (21) combined with Eqs. (12) and (13) results in the following differential equation of the radial displacement:

$$\frac{du}{dr} + k\omega\frac{u}{r} + D_1\left(\sigma_\theta^{*res} - P_0^{*res}\right) + D_2\left(\sigma_\theta^{*res}\right)^a = 0 \tag{23}$$

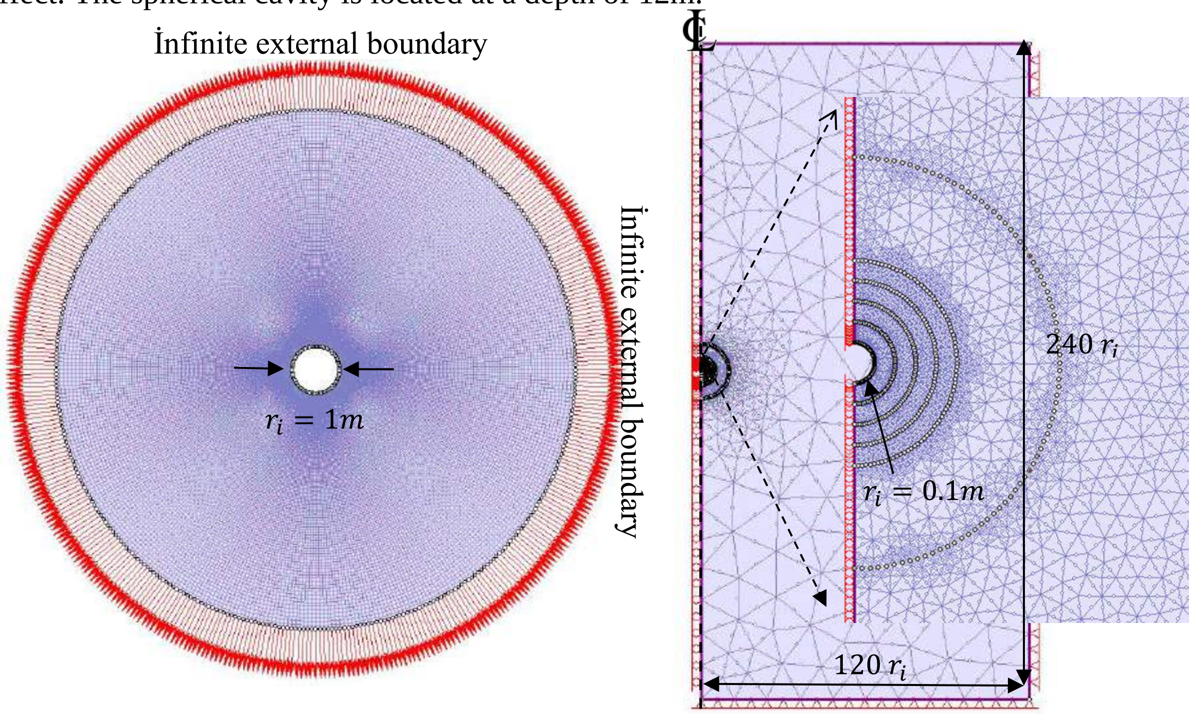
where D_1 and D_2 are two dimensionless coefficients defined respectively as $\frac{(1+k\omega)(1-2\nu)}{2I_r(1+\nu)^{k-1}}$ and $\frac{(1+\nu(k-2)-\nu k\omega)}{2I_r(1+\nu)^{k-1}}$. It is important to mention that the rigidity index is expressed this time as a function of the residual shear modulus as $I_r = G^{res}/(\sigma_{ci}(m_b^{res})^\beta)$. Knowing the displacement at the (EPB), the solution of the above differential equation is expressed as:

$$u(r) = r^{-k\omega}\left[u_{EPB}r_p^{k\omega} + \int_r^{r_p} \rho^{k\omega}(D_1(\sigma_\theta^* - P_0^*) + D_2(\sigma_\theta^*)^a)\, d\rho\right] \tag{24}$$

Figure 4 shows the evolution of $u(r)$ for both cylindrical and spherical cavities.

5. VALIDATION

Using the finite element software RS² 9 Modeler [8], a finite element analysis is conducted to validate the analytical results. The geometry of the models as well as the selected boundary conditions for both cylindrical and spherical cavities are as shown in Figure 5. A plane strain conditions are assumed for the cylindrical cavity with an internal radius of $1m$ and default external boundary set as infinite. On the other hand, a spherical cavity with an internal radius $r_i = 0.1\ m$ is modeled with a rock domain broad enough $(120r_i \times 240r_i)$ to prevent boundary effect. The spherical cavity is located at a depth of 12m.



(a) Cylindrical cavity (b) Spherical cavity

***Figure 5**: Geometry and boundary conditions of the problem*

A distributed load normal to the cylindrical and the spherical cavity cluster of 5 $[MPa]$ and 10 $[MPa]$, respectively is prescribed. İt should be emphasized that the spherical cavity expansion is a relatively hard boundary value problem which consumes a significant amount of computational power when solved using the finite element method.

The rock mass in the FEM model has properties similar to that of an undisturbed claystone ($D = 0$) with: $\sigma_{ci} = 35$ [MPa], $m_i = 4$ and $MR = E_i/\sigma_{ci} = 250$ and $GSI = 80$. The peak and residual deformation modulus are evaluated based on H-B core parameters as $7703\ [MPa]$ and $4715\ [MPa]$, respectively. These values can be simply estimated from the software library [8]. Since the Poisson's ratio ν does not usually affect rock behavior in a significant manner, a standard value equals to 0.3 is used. Note that the dilatancy angle ψ as well as the far field pressure P_0 are set equal to zero. As shown in Figure 4, analytical solutions are in excellent agreement with the FEM predictions emphasizing the accuracy of the developed solutions.

6. CONCLUSION

The expansion of cylindrical and spherical cavities in an infinite medium consisting of an elastic – brittle plastic Hoek-Brown material is investigated. The use of a scaled form of the H-B failure criterion leads to considerable simplifications in defining the elastoplastic response of the rock mass. Analytical expressions are obtained for the stress and displacement field. Although the solutions require some numerical integrations over the plastic region, they have the advantage of being highly robust considerably in comparison with other numerical techniques such as the finite element method. The analytical expressions were validated employing the finite element method.

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